

MATH 2050 - Cauchy sequences

(Reference: Bartle § 3.5)

Q: When is (x_n) convergent (without knowing its limit)?

A1: "MCT" bdd + monotone \Rightarrow convergent

BUT " \Leftarrow " is FALSE Ex: $(x_n) = \left(\frac{(-1)^n}{n}\right) \rightarrow 0$

A2: "Cauchy" \Leftrightarrow convergent
"iff"

Defⁿ: A seq. (x_n) is called Cauchy if

$\forall \varepsilon > 0, \exists H = H(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq H.$$



Remark: Compared to the ε - K defⁿ for convergence of (x_n) , we DO NOT need to refer the potential limit x .

Example 1: $(x_n) := \left(\frac{1}{n}\right)$ is Cauchy. (Also $\left(\frac{1}{n}\right) \rightarrow 0$)

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \frac{2}{\varepsilon}$.

Then, $\forall n, m \geq H$,

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H} < \varepsilon$$

Example 2: $(x_n) := (1 + (-1)^n)$ is NOT Cauchy

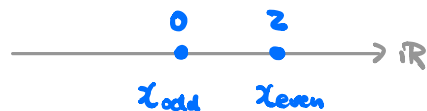
Pf: n odd : $x_n = 1 - 1 = 0$

n even : $x_n = 1 + 1 = 2$

$(x_n) = (0, 2, 0, 2, 0, 2, \dots)$

divergent!

Let $\epsilon_0 = 1 > 0$. For any $H \in \mathbb{N}$ fixed.



\exists odd $m \geq H$

\exists even $n \geq H$

st. $|x_n - x_m| = |2 - 0| = 2 \geq 1 = \epsilon_0$

nec. & suff. condition

Thm: (x_n) convergent $\Leftrightarrow (x_n)$ Cauchy

Proof: " \Rightarrow " Assume (x_n) is convergent, say $\lim(x_n) = x$.

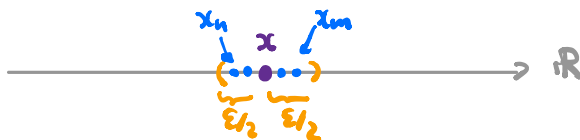
By defⁿ, let $\epsilon > 0$ be given, then $\exists K = K(\frac{\epsilon}{2}) \in \mathbb{N}$ st.

$|x_n - x| < \epsilon/2 \quad \forall n \geq K$ (*)

Choose $H = K \in \mathbb{N}$. Then $\forall n, m \geq H = K$,

$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (*)

So, (x_n) is Cauchy.



" \Leftarrow " Assume (x_n) is Cauchy.

Claim 1: (x_n) is bdd

Pf of Claim: Since (x_n) is Cauchy, take $\epsilon_0 = 1 > 0$, then

$\exists H = H(1) \in \mathbb{N}$ st. $\forall n, m \geq H$,

$|x_n - x_m| < 1 = \epsilon_0$

Fix $m = H$, then by reverse Δ -ineq and the above.

$$| |x_n| - |x_H| | \leq |x_n - x_H| < 1 \quad \forall n \geq H$$

$$\Rightarrow |x_n| \leq |x_H| + 1 \quad \forall n \geq H$$

Take $M := \max \{ |x_1|, \dots, |x_{H-1}|, |x_H| + 1 \}$

Then, $|x_n| \leq M, \forall n \in \mathbb{N}$. ie (x_n) is bdd.

Claim 2: (x_n) is convergent

Pf of claim: Since (x_n) is bdd by Claim 1,

"BWT" $\Rightarrow \exists$ convergent subseq. $(x_{n_k}) \rightarrow x \in \mathbb{R}$.

potential
candidate
for our limit

Want to show: $(x_n) \rightarrow x$

By Cauchy defⁿ, let $\varepsilon > 0$ be fixed but arbitrary.

then $\exists H = H(\frac{\varepsilon}{2}) \in \mathbb{N}$ st.

$$|x_m - x_n| < \frac{\varepsilon}{2} \quad \forall n, m \geq H \quad \text{--- (*)}$$

Since the subseq. $(x_{n_k}) \rightarrow x$ as $k \rightarrow \infty$, by defⁿ.

$\exists K = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ st

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \quad \forall k \geq K \quad \text{--- (**)}$$

Fix a $k \geq K$ st $n_k \geq H$.

Then, $\forall n \geq H$, we have

$$|x_n - x| \leq \underbrace{|x_n - x_{n_k}|}_{(*)} + \underbrace{|x_{n_k} - x|}_{(**)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Example: Let (x_n) be the sequence defined by

$$x_1 := 1 \quad ; \quad x_2 := 2 \quad ; \quad x_n := \frac{1}{2}(x_{n-1} + x_{n-2}) \quad \forall n \geq 3.$$

Show that (x_n) is convergent and find $\lim (x_n)$.

Think: $(x_n) := (1, 2, 1.5, 1.75, 1.625, \dots)$

bdd, NOT monotone,

Pf: By M.I. (Exercise), we have

$$\bullet \quad 1 \leq x_n \leq 2 \quad \forall n \in \mathbb{N}$$

$$\bullet \quad |x_{n+1} - x_n| = \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$$

Claim: (x_n) is "Cauchy"

Pf of Claim: Let $\varepsilon > 0$ be fixed but arbitrary.

$$\text{Choose } H \in \mathbb{N} \text{ s.t. } H > \frac{4}{\varepsilon}.$$

Then, $\forall m, n \geq H$, we want to show

$$|x_m - x_n| < \varepsilon \quad \forall m, n \geq H$$

W.L.O.G. assume $m > n \geq H$.

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$= \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \dots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{m-n-1}} \right)$$

$$< \frac{1}{2^{n-1}} \cdot 2 = \frac{1}{2^{n-2}} \leq 4 \cdot \frac{1}{2^H} \leq 4 \cdot \frac{1}{H} < \varepsilon$$

By Cauchy Criteria, $\lim (x_n) =: x$ exists.

Consider the subseq. $(x_{2k-1})_{k \in \mathbb{N}}$

Note: $\lim_{k \rightarrow \infty} (x_{2k-1}) = x$

$$x_{2k-1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots + \frac{1}{2^{2k-3}}$$

$$= 1 + \frac{1/2 (1 - \frac{1}{4^{k-1}})}{1 - \frac{1}{4}}$$

Take $k \rightarrow \infty$. we have $x = 1 + \frac{1/2}{3/4} = \frac{5}{3} \neq$

...

$$x_n = \frac{1}{2} (x_{n-1} + x_{n-2})$$

Take $n \rightarrow \infty$,

$$x = \frac{1}{2} (x + x) = x$$

↑
not helpful